LIII. *The Refolution of a General Propofition for Determining the* horary *Alteration of the Pofition of the Terreftrial Equator, from the Attraction of the Sun and Moon: With fome Remarks on the Solutions given by other Authors to that difficult and important Problem. By Mr.* Tho. Simpfon, *F. R. S*

Read Dec. 22, 1757.

SINCE the time, that that excellent Aftronomer, my much honoured friend Dr. Bradley, publifhed his obfervations and difcoveries concerning the inequalities of the preceffion of the equinox, and of the obliquity of the ecliptic, depending on the pofition of the lunar nodes, mathematicians, in different parts of Europe, have fet themfelves diligently to compute, from phyfical principles, the effects produced by the fun and moon, in the pofition of the terreftrial equator; and to examine whether thefe effects do really correfpond with the obfervations

Two papers on this fubject have already appeared in the Philofophical Tranfactions; in which the authors have fhewn evident marks of fkill and penetration. There is, neverthelefs, one part of the fubject, that feems to have been paffed over without a due degree of attention, as well by both thofe gentlemen, as by Sir Ifaac Newton himfelf.

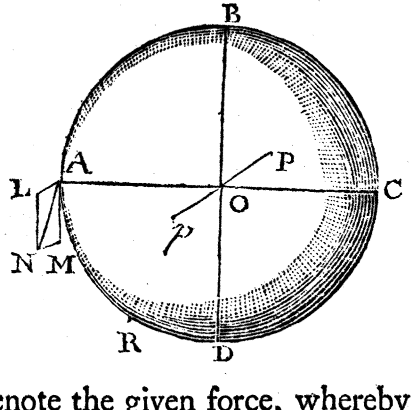
This part, which, upon account of phyfical difficulties, is indeed fomewhat flippery and perplexing, I fhall make the principal fubject of this effay.

GENE-

GENERAL PROPOSITION.

Suppoſing an homogeneous ſphere OABCD (Fig. 1.) *revolving uniformly about its centre, to be acted on at the extremity* A *of the radius* OA, *in a direction* AL *perpendicular to the plane of the equator* ABCD, *and parallel to the axis of rotation* Pp, *by a given force, tending to generate a new motion of rotation at right angles to the former; It is propoſed to determine the change, that will ariſe in the direction of the rotation in conſequence of the ſaid force.*

FIG. 1.



Let F denote the given force, whereby the motion about the axis Pp is diſturbed, ſuppoſing f to repreſent the centrifugal force of a ſmall particle of matter in the circumference of the equator, ariſing from the ſphere's rotation; and let the whole number of ſuch particles, or the content of the ſphere, be denoted by c: let alſo the momentum of rotation of the whole ſphere, or of all the particles, be ſuppoſed, in proportion to the momentum of an equal number of particles, revolving at the diſtance O A of the remoteſt point A, as n is to *unity*.

It is well known, that the centripetal force, whereby any body is made to revolve in the circumference of a circle, is ſuch, as is ſufficient to generate all the motion in the body, in a time equal to *that*, wherein the body deſcribes an arch of the circumference, equal in length to the radius. Therefore, if we here take the arch A R = O A, and aſſume m to expreſs the time, in which that arch would be uniformly deſcribed by the point A, the *motion* of a particle of matter at A (whoſe central force is repreſented by f) will be equal to *that*, which might be uniformly generated by the force f, in the time m; and the motion of as many particles (revolving, all, at the ſame diſtance) as are expreſſed by cn (which, by hypotheſis, is equal to the momentum of the whole body), will, conſequently, be equal to the momentum, that might be generated by the force $f \times cn$, in the ſame time m. Whence it appears, that the momentum of the whole body about its axe Pp is in proportion to the momentum generated in a given particle of time m', by the given force F in the direction A L, as $ncf \times m$ is to $F \times m'$, or, as *unity* to $\frac{F}{ncf} \times \frac{m'}{m}$ (becauſe the quantities of motion produced by unequal forces, in unequal times, are in the ratio of the forces and of the times, conjunctly). Let, therefore, A L be taken in proportion to A M, as $\frac{F}{ncf} \times \frac{m'}{m}$ is to *unity* (ſuppoſing A M to be a tangent to the circle A B C D in A), and let the parallelogram A M N L be compleated; drawing alſo the diagonal A N; then, by the

the compofition of forces, the angle NAM (whofe tangent, to the radius OA, is expreffed by $OA \times \frac{F}{ncf} \times \frac{m'}{m}$) will be the change of the direction of the rotation, at the end of the aforefaid time (m'). But, this angle being exceeding fmall, the tangent may be taken to reprefent the meafure of the angle itfelf; and, if Z be affumed to reprefent the arch defcribed by A, in the fame time (m') about the center O, we fhall alfo have $\frac{m'}{m} = \frac{Z}{AR} = \frac{Z}{AO}$, and confequently $OA \times \frac{F}{ncf} \times \frac{m'}{m} = Z \times \frac{F}{ncf}$. From whence it appears, that the angle expreffing the change of the direction of the rotation, during any fmall particle of time, will be in proportion to the angle defcribed about the axe of rotation in the fame time, as $\frac{F}{ncf}$ is to *unity*. *Q. E. I.*

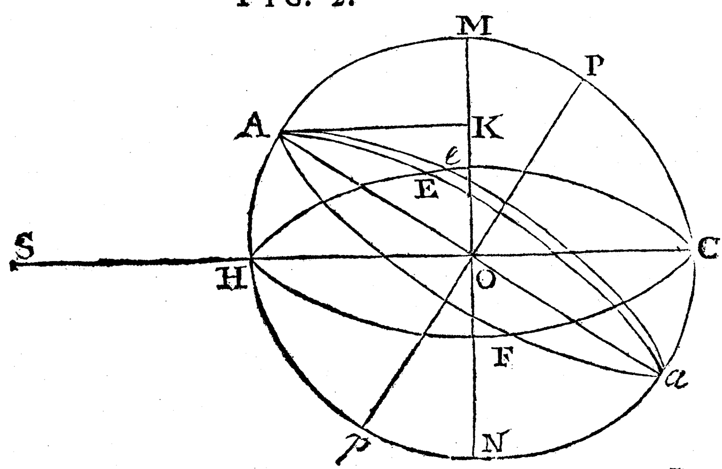
Altho', in the preceding propofition, the body is fuppofed to be a perfect fphere, the folution, neverthelefs, holds equally true in every other fpecies of figures, as is manifeft from the inveftigation. It is true, indeed, that the value of n will not be the fame in thefe cafes, even fuppofing thofe of c, f and F to remain unchanged; except in the fpheroid only, where, as well as in the fphere, n will be $= \frac{2}{5}$; the momentum of any fpheroid about its axis being 2-5ths of the momentum of an equal quantity of matter placed in the circumference of the equator, as is very eafy to demonftrate.

But to fhew now the ufe and application of the general proportion here derived, in determining the regrefs of the equinoctial points of the terreftrial

 fpheroid,

ſpheroid, let A E a F (*Fig.* 2.) be the equator, and P p the axis of the ſpheroid: alſo let H E C F repreſent the plane of the ecliptic, S the place of the ſun, and H A P N H the plane of the ſun's declination, making right-angles with the plane of the equator A E a F: then, if AK be ſuppoſed parallel, and O K M perpendicular, to O S, and there be aſſumed T and t to expreſs the reſpective times of the annual and diurnal revolutions of the earth, it will appear (from the *Principia*, B. III. prop. xxv.) that the force, with which a particle of matter at A tends to recede from the line O M in conſequence of the ſun's attraction, will be expreſſed by $\frac{3tt}{TT}\times\frac{AK}{OA}\times f$; f denoting the centrifugal force of the ſame particle, ariſing from the diurnal rotation. Hence, by the reſolution of forces, $\frac{3tt}{TT}\times\frac{AK}{OA}\times\frac{OK}{OA}\times f$ will be the effect of that particle, in a direction perpendicular to O A, to turn the earth about its center O.

FIG. 2.



But

But it is demonſtrated by Sir Iſaac Newton, and by other authors, that the force of all the particles, or of all the matter in the whole ſpheroid $APap$, to turn *it* about its center, is equal to $\frac{1}{5}$th of the force of a quantity of matter, placed at A, equal to the exceſs of the matter in the whole ſpheroid above *that* in the inſcribed ſphere, whoſe axis is Pp. Now this exceſs (aſſuming the ratio of π to 1, to expreſs *that* of the area of a circle to the ſquare of the radius) will be truly repreſented by $\frac{4\pi}{3} \times OP \times \overline{OA^2 - OP^2}$; and, conſequently, the force of all the matter in the whole earth, by $\frac{3tt}{TT} \times \frac{AK}{OA} \times \frac{OK}{OA} \times \frac{4\pi}{15} \times OP \times \overline{OA^2 - OP^2}$. Let, therefore, this quantity be now ſubſtituted for F, in the general formula $\frac{F}{ncf}$, writing, at the ſame time, $\frac{4\pi}{3} \times OA^2 \times OP$, and $\frac{2}{5}$, in the place of their equals c and n; by which means we have (here) $\frac{F}{ncf} = \frac{3tt}{2TT} \times \frac{OA^2 - OP^2}{OA^2} \times \frac{AK \times OK}{OA^2}$. Put the given quantity $\frac{3tt}{2TT} \times \frac{OA^2 - OP^2}{OA^2} = k$; and let the angle EAe repreſent the horary alteration of the poſition of the terreſtrial equator, ariſing from the force F (here determined), and let the arch Ee be the regreſs of the equinoctial point E, correſponding thereto: then, in the triangle EAe (conſidered as ſpherical) it will be ſin. e : ſin. AE (: : ſin. EAe : ſin. Ee) : : EAe : Ee $\left(= \frac{\text{ſin. AE} \times EAe}{\text{ſin. E}}\right) = k \times \frac{\text{ſin. AE}}{\text{ſin. E}} \times \frac{AK \times OK}{OA^2} = k \times \frac{\text{ſin. AE} \times \text{coſ. AH} \times \text{ſin. AH}}{\text{ſin. E}}$. But in the triangle EHA, right-angled at A (where HA is ſuppoſed

ſuppoſed to repreſent the ſun's declination, AE his right aſcenſion, and HE his diſtance from the equinoctial point E *) we have *(per ſpherics)*

$$\text{ſin. } AE : 1 \text{ (rad.)} :: \text{co-t. } E : \text{co-t. } AH,$$

$$\overline{\text{ſin. } AH}^2 : \overline{\text{ſin. } EH}^2 :: \overline{\text{ſin. } E}^2 : 1^2 (\overline{\text{rad.}}^2)$$

From whence we get, $\text{ſin. } AE \times \text{co-t. } AH \times \overline{\text{ſin. } AH}^2 = \overline{\text{ſin. } EH}^2 \times \text{co-t. } E \times \overline{\text{ſin. } E}^2$. But $\text{co-t. } AH \times \text{ſin. } AH = \text{co-ſ. } AH \times 1 \text{ (rad.)}$, and $\text{co-t. } E \times \text{ſin. } E = \text{co-ſ. } E \times 1 \text{ (rad.)}$: therefore $\text{ſin. } AE \times \text{co-ſ. } AH \times \text{ſin. } AH = \overline{\text{ſin. } EH}^2 \times \text{co-ſ. } E \times \text{ſin. } E$; and, conſequently, $k \times \frac{\text{ſin. } AE \times \text{co-ſ. } AH \times \text{ſin. } AH}{\text{ſin. } E} = k \times \text{co-ſ. } E \times \overline{\text{ſin. } EH}^2$ $(= Ee)$.

Let, now, the ſun's longitude EH be denoted by Z (conſidered as a flowing quantity); then, $\overline{\text{ſin. } Z}^2$ being $= \frac{1}{2} - \frac{1}{2} \text{ co-ſ. } 2Z$, we ſhall have $k \times \text{co-ſ. } E \times \overline{\text{ſin. } EH}^2 = \frac{1}{2} k \times \text{co-ſ. } E \times \overline{1 - \text{co-ſ. } 2Z}$. But the angle deſcribed about the axe of rotation Pp, in the time that the ſun's longitude is augmented by the particle $\dot{Z}$, will be $= \frac{T}{t} \times \dot{Z}$. Therefore (by the general propoſition) we have, as $1 : \frac{1}{2} k \times \text{co-ſ. } E \times \overline{1 - \text{co-ſ. } 2Z} :: \frac{T}{t} \times \dot{Z} : \frac{1}{2} k \times \frac{T}{t} \times \text{co-ſ. } E \times \dot{Z} - \dot{Z} \text{ co-ſ. } 2Z$, the true regreſs of the equinoctial point E, during that

* No error ariſes from conſidering the triangles EAe and AEH, as being formed on the ſurface of a ſphere, tho' the earth itſelf is not accurately ſuch. The angle (EAa) repreſenting the effect of the ſolar force, is properly referred to the ſurface of a ſphere; therefore (after the meaſure thereof is truly determined) the figure $APap$ is itſelf taken as a ſphere, in order to avoid the trouble of introducing a new ſcheme.

that time: whose fluent, $\frac{1}{2} k \times \frac{T}{t} \times$ co-s. E $\times \overline{Z - \frac{1}{2} \text{sin. } 2 Z}$, will consequently be the total regress of the point E, in the time that the sun, by his apparent motion, describes the arch HE or Z; which, on the sun's arrival at the solstice, becomes barely $= \frac{1}{2} k \times \frac{T}{t} \times$ co-s. E $\times$ an arch of 90°: the quadruple whereof, or $\frac{1}{2} k \times \frac{T}{t} \times$ co-s. E $\times$ 360° $\left(= \frac{3t}{4T} \times \frac{OA^2 - OP^2}{OA^2}\right.$ $\times$ co-s. E $\times$ 360°) is therefore the whole annual precession of the equinox caused by the sun. This, in numbers (taking $\frac{OP}{OA} = \frac{229}{230}$) comes out $\frac{3}{4 \times 366\frac{1}{4}} \times \frac{2}{230\frac{1}{2}} \times 0.917176 \times 360° = 21'' \, 6'''$.

The very ingenious M. Silvabelle, in his essay on this subject, inserted in the 48th volume of the Philosophical Transactions, makes the quantity of the annual precession of the equinox, caused by the sun, to be the half, only, of what is here determined. But this gentleman appears to have fallen into a twofold mistake. First, in finding the *momenta of rotation* of the terrestrial spheroid, and of a very slender ring, at the equator thereof; which *momenta* he refers to an axis perpendicular to the plane of the sun's declination, instead of the proper axe of rotation, standing at right angles to the plane of the equator. The difference, indeed, arising from thence, with respect to the spheroid (by reason of its near approach to a sphere) will be inconsiderable; but, in the ring, the case will be quite otherwise; the equinoctial points thereof being made to recede just twice as fast as they ought to do. This may ſeem the more ſtrange, if regard be had to the concluſions, relating to the nodes of a ſatellite, derived from this very aſſumption. But, that theſe concluſions are true, is owing to a ſecond, or ſubſequent miſtake, at Art. 27; where the meaſure of the ſun's force is taken the half, only, of the true value; by means whereof the motion of the equinoctial points of the ring is reduced to its proper quantity, and the motion of the equinoctial points of the terreſtrial ſpheroid, to the half of what it ought to be.

That expert geometrician M. Cha. Walmſley, in his Eſſay on the Preceſſion of the Equinox, printed in the laſt volume of the Philoſophical Tranſactions, has judiciouſly avoided all miſtakes of this laſt kind, reſpecting the ſun's force, by purſuing the method, pointed out by Sir Iſaac Newton; but, in determining the effect of that force, has fallen into others, not leſs conſiderable than thoſe above adverted to.

In his third Lemma, the momentum of the whole Earth, about its diameter, is computed on a ſuppoſition, that the momentum or force of each particle is proportional to its diſtance from the axis of motion, or barely as the quantity of motion in ſuch particle, conſidered abſtractedly. No regard is, therefore, had to the lengths of the unequal levers, whereby the particles are ſuppoſed to receive and communicate their motion: which, without doubt, ought to have been included in the conſideration.

In his firſt propoſition, he determines, in a very ingenious and conciſe manner, the true annual motion of the nodes of a ring (or of a ſingle ſatellite) at the earth's equator, revolving with the earth itſelf, about

its

its center, in the time of one fiderial day. This motion he finds to be $= \frac{3 \text{ co-f. } 23^\circ 29'}{4 \text{ rad.}} \times \frac{1}{366\frac{1}{4}} \times 360^\circ$. Then, in order to infer from thence, the motion of the equinoctial points of the earth itfelf, he, firft, diminifhes that quantity, in the ratio of 2 to 5: Becaufe (as is demonftrated by Sir Ifaac Newton in his 2d Lemma) the whole force of all the particles fituated without the furface of a fphere, infcribed in the fpheroid, to turn the body about its center, will be only 2-5ths of the force of an equal number of particles uniformly difpofed round the whole circumference of the equator, in the fafhion of a ring. The quantity $\left(\frac{3 \text{ co-f. } 23^\circ 29'}{4 \text{ rad.}} \times \frac{2}{5} \times \frac{1}{366\frac{1}{4}} \times 360^\circ\right)$ thus arifing, will, therefore, exprefs the true motion of the equinoctial points of a ring, equal in quantity of matter to the excefs of the whole earth above the infcribed fphere, when the force whereby the ring tends to turn about its diameter is fuppofed equal to the force whereby the earth itfelf tends to turn about the fame diameter, in confequence of the fun's attraction. Thus far our author agrees with Sir Ifaac Newton; but, in deriving from hence the motion of the equinoctial points of the earth itfelf, he differs from him; and, in the corollary to his third Lemma, affigns the reafons, why he thinks Sir Ifaac Newton, in this particular, has *wandered a little from the truth.* Inftead of diminifhing the quantity above exhibited (as Sir Ifaac has done) in the ratio of all the motion in the ring to the motion in the whole earth, he diminifhes it in the ratio of the motion of all the matter above the furface of the infcribed fphere to the motion of the whole earth: which matter, tho' equal to that of the ring, has neverthelefs a different momentum, arifing from the different fituation of the particles in refpect to the axis of motion.

But fince the aforefaid quantity, from whence the motion of the earth's equinox is derived, as well by this gentleman, as by Sir Ifaac Newton, expreffes truly the annual regrefs of the equinoctial points of the ring (and not of the hollow figure formed by the faid matter, which is greater, in the ratio of 5 to 4) it feems, at leaft, as reafonable to fuppofe, that the faid quantity, to obtain from thence the true regrefs of the equinoctial points of the earth, ought to be diminifhed in the former of the two ratios above fpecified, as that it fhould be diminifhed in the latter. But, indeed, both thefe ways are defective, even fuppofing the momenta to have been truly computed; the ratio, that ought to be ufed here, being that of the momenta of the ring and earth about the proper axe of rotation of the two figures, ftanding at right-angles to the plane of the ring and of the equator. Now this ratio, by a very eafy computation, is found to be as $\overline{230}^2 - \overline{229}^2$ to $\frac{2}{5}$ of $\overline{230}^2$; whence the quantity fought comes out $= \frac{3 \text{ co-f. } 23^\circ 29'}{4 \text{ rad.}} \times \frac{1}{366\frac{1}{4}} \times \frac{\overline{230}^2 - \overline{229}^2}{230^2} \times 360^\circ = 21'' 6'''$: which is the fame that we before found it to be, and the double of what this author makes it.

What has been faid hitherto, relates to that part of the motion only, arifing from the force of the fun. It will be but juftice to obferve here, that the effect of the moon, and the inequalities depending on the pofition of her nodes, are truly affigned by both the gentlemen

gentlemen above-named; the ratio of the diameters of the earth, and the denſity of the moon being ſo aſſumed, as to give the maxima of thoſe inequalities, ſuch as the obſervations require: in conſequence whereof, and from the law of the increaſe and decreaſe (which is rightly determined by theory, tho' the abſolute quantity is not) a true ſolution, in every other circumſtance, is obtained.

The freedom, with which I have expreſſed myſelf, and the liberty I have here taken, to animadvert on the works of men, who, in many places, have given inconteſtible proofs of ſkill and genius, may, I fear, ſtand in need of ſome apology. 'Tis poſſible I may be thought too peremptory. Indeed, I might have delivered my ſentiments with more caution and addreſs: but, had not I imagined myſelf quite clear in what has been advanced, from a multitude of concurrent reaſons, I ſhould have thought it too great a preſumption to have ſaid any thing at all here, on this ſubject. The great regard I have for this Society, of which I have the honour to be a member, will, I hope, be conſidered as the motive for my having attempted to rectify ſome overſights, that have occurred in the works of this learned body.